

SIMPLICIAL INVERSE SEQUENCES IN EXTENSION THEORY

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ABSTRACT. In extension theory, in particular in dimension theory, it is frequently useful to represent a given compact metrizable space X as the limit of an inverse sequence of compact polyhedra. We are going to show that, for the purposes of extension theory, it is possible to replace such an X by a better metrizable compactum Z . This Z will come as the limit of an inverse sequence of triangulated polyhedra with simplicial bonding maps that factor in a certain way. There will be a cell-like map $\pi : Z \rightarrow X$, and we shall show that if K is a CW-complex with $X\tau K$, then $Z\tau K$.

1. INTRODUCTION

In extension theory, and in particular the theories of dimension and cohomological dimension \dim_G over an abelian group G ([Ku]), it is frequently useful to represent a given compact metrizable space X as the limit of an inverse sequence of compact polyhedra. This was of importance in the proofs of the Edwards-Walsh cell-like resolution theorem ([Ed], [Wa]), Dranishnikov's \mathbb{Z}/p -resolution theorem ([Dr]), and Levin's \mathbb{Q} -resolution theorem ([Le]). In each case the hypothesis was that $\dim_G X \leq n$ (the abelian group G depending on which of the three cases was under consideration), and the first step in their respective proofs of the existence of a resolution of the desired type (cell-like, \mathbb{Z}/p -acyclic, \mathbb{Q} -acyclic, respectively) was to represent X as the inverse limit of an inverse sequence of compact polyhedra. That this can always be done comes originally from H. Freudenthal ([Fr]), but the result can be found also as Corollary 4.10.11. in [Sa]. It stipulates that each compact metrizable space can be written as the inverse limit of an inverse sequence (X_i, p_i^{i+1}) of finite polyhedra with surjective piecewise linear bonding maps $p_i^{i+1} : X_{i+1} \rightarrow X_i$, where piecewise linear means that the domain and range of p_i^{i+1} can be triangulated in such a manner that p_i^{i+1} is simplicial with respect to these triangulations.

One might ask if it is possible to arrange such an inverse sequence so that each polyhedron has a fixed triangulation and so that all the bonding maps are simplicial with respect to these triangulations. It was shown in [Ma] that this is not always attainable. On the other hand, at least for the purposes of extension theory, can such an obstacle be removed? Let us give a brief explanation of what we have accomplished in this direction.

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When X and K are spaces, we are going to write $X\tau K$ to mean that X is an absolute co-extensor for K , i.e., for each closed subset A of X and map $f : A \rightarrow K$, there exists a map $g : X \rightarrow K$ that extends f . This is the fundamental notion of extension theory, and typically K is a CW-complex. Let X be a compact metrizable space. Then $X\tau S^n$ if and only if $\dim X \leq n$. For cohomological dimension \dim_G over an abelian group G , one has that $\dim_G X \leq n$ if and only if $X\tau K(G, n)$ where $K(G, n)$ is any Eilenberg-MacLane complex of type (G, n) . Thus extension theory is a unifying structure in the study of dimension theory.

Our main result is Theorem 8.5. It states that for a given nonempty metrizable compactum X , there exists a metrizable compactum Z and a cell-like map (see Definition 8.3) $\pi : Z \rightarrow X$ such that if K is a CW-complex with $X\tau K$, then $Z\tau K$. Moreover, the compactum Z comes as the limit of an inverse sequence $\mathbf{Z} = (|T_j|, g_j^{j+1})$ in which all the bonding maps are simplicial with respect to the given finite triangulations T_i of the polyhedra $|T_i|$. These g_j^{j+1} have simplicial factorizations (see Lemma 5.2) $g_j^{j+1} = f_j^{j+1} \circ \varphi_{j+1} : |T_{j+1}| \rightarrow |T_j|$, $\varphi_{j+1} : |T_{j+1}| \rightarrow |\tilde{T}_{j+1}|$, where T_{j+1} is a subdivision of \tilde{T}_{j+1} and φ_{j+1} is a simplicial approximation to the identity map; these play a prominent role in our development.

We also provide a theory of “adjustments” (Definition 7.3) to such a \mathbf{Z} . If $n \geq 0$, (j_i) is an increasing sequence in \mathbb{N} , and each $g_{j_i}^{j_{i+1}} : |T_{j_{i+1}}^{(n)}| \rightarrow |T_{j_i}^{(n)}|$ is replaced by a map h_i^{i+1} that is a T_{j_i} -modification of it (see Definition 2.1), then we get a new inverse sequence $\mathbf{M} = (|T_{j_i}^{(n)}|, h_i^{i+1})$. There is a uniquely induced surjective map $\pi : \lim \mathbf{M} \rightarrow X$. This is covered in Lemma 7.8, where it is shown how to describe each fiber $\pi^{-1}(x)$ of π as the limit of three different sub-inverse sequences of \mathbf{M} . This was employed in our proof of Theorem 8.5. The existence of such maps whose fibers are so well-described has the potential to be used in other resolution theorems.

2. SIMPLICIAL COMPLEXES AND EXTENSORS

For each simplicial complex T , $|T|$ will designate its polyhedron with the weak topology. The n -skeleton of T is going to be written $T^{(n)}$. If T is finite, then we shall supply it with the metric induced by T ; in this case the weak topology on $|T|$ is the same as the metric topology. If $v \in T^{(0)}$, then $\text{st}(v, T)$ will denote the *open star* of v in T and $\overline{\text{st}}(v, T)$ will denote the *closed star* of v in T . Of course, $\text{st}(v, T)$ is an open neighborhood of v in the polyhedron $|T|$, $\overline{\text{st}}(v, T)$ is a closed subset of $|T|$, and $\text{st}(v, T) \subset \overline{\text{st}}(v, T)$. Moreover, each of $\text{st}(v, T)$ and $\overline{\text{st}}(v, T)$ is contractible, and there is a unique subcomplex $S_{v,T}$ of T such that $|S_{v,T}| = \overline{\text{st}}(v, T)$. We make the convention that $T^{(\infty)} = T$. Map will always mean continuous function. Also,

Definition 2.1. *If $f : X \rightarrow |T|$ is a function where T is a simplicial complex, then a function $g : X \rightarrow |T|$ is called a T -**modification** of f if for each $x \in X$ and simplex σ of T with $f(x) \in \sigma$, $g(x) \in \sigma$. This is equivalent to saying that for each $x \in X$ and simplex σ of T with $f(x) \in \text{int}(\sigma)$, $g(x) \in \sigma$.*

The “straight line” homotopy gives us the next fact.

Lemma 2.2. *Let T be a finite simplicial complex, X a space, and f, g maps of X to $|T|$ such that g is a T -modification of f . Then $g \simeq f$. \square*

Definition 2.3. For each simplicial complex T and nonempty subset $D \subset |T|$, we shall denote by $N_{D,T}$ the subcomplex of T consisting of the simplexes of T that intersect D and all faces of such simplexes. This is the **simplicial neighborhood** of D in T .

Let us review the notion of *extensor* ([Hu]). A space K is an *absolute neighborhood extensor* for a space X , written $K \in \text{ANE}(X)$, if each map of a closed subspace A of X to K extends to a map of a neighborhood of A in X to K . An ANR, absolute neighborhood retract, is a metrizable space that is an absolute neighborhood extensor for any metrizable space.

We state a version of (R1) from page 74 of [MS] that will be suitable for our purposes.

Lemma 2.4. Let $\mathbf{Y} = (Y_i, g_i^{i+1})$ be an inverse sequence of metrizable compacta, $Y = \lim \mathbf{Y}$, P an ANR with metric d , $\mu : X \rightarrow P$ a map, and $\epsilon > 0$. Then there exists i such that for all $j \geq i$, there is a map $f : Y_j \rightarrow P$ with $d(f \circ g_{j,\infty}, \mu) < \epsilon$.

Lemma 2.5. Let K be a CW-complex. Then K has an open cover \mathcal{V} such that any two \mathcal{V} -close maps of any space to K are homotopic.

Proof. There exists a simplicial complex L such that $|L|_m$, that is $|L|$ with the metric topology, is homotopy equivalent to K . Choose a homotopy equivalence $f : K \rightarrow |L|_m$. By Theorem III.11.3 (page 106) of [Hu], $|L|_m$ is an ANR. Theorem IV.1.1 (page 111) of [Hu] shows that there is an open cover \mathcal{W} of $|L|_m$ having the property that any two \mathcal{W} -close maps of any space to $|L|_m$ are homotopic. The open cover needed for K is $\mathcal{V} = f^{-1}(\mathcal{W})$. \square

There is a relatively standard technique that can be used to help detect when the limit of an inverse sequence of metrizable compacta is an absolute co-extensor for a given CW-complex. Here are the needed concepts.

Proposition 2.6. Let $\mathbf{Z} = (Z_i, g_i^{i+1})$ be an inverse sequence of nonempty compact metrizable spaces, $Z = \lim \mathbf{Z}$, and K a CW-complex. Suppose that for each $i \in \mathbb{N}$, closed subset D of Z_i , and map $f : D \rightarrow K$, there exist $j \geq i$ and a map $g : Z_j \rightarrow K$ such that for all $x \in (g_i^j)^{-1}(D)$, $g(x) = f \circ g_i^j(x)$. Then $Z \tau K$. \square

Definition 2.7. Let Z be a nonempty space and \mathcal{B} a collection of nonempty closed subsets of Z . We shall call \mathcal{B} a **base** for the closed subsets of Z provided that for each closed subset C of Z and neighborhood U of C in Z , there exists $A \in \mathcal{B}$ with $C \subset \text{int}_Z A \subset A \subset U$.

Lemma 2.8. Every compact metrizable space has a countable base for its closed subsets. \square

Definition 2.9. For each compact metrizable space X , let $\mathcal{B}(X)$ designate a fixed countable base for the closed subsets of X .

Proposition 2.10. Let $\mathbf{Z} = (Z_i, g_i^{i+1})$ be an inverse sequence of compact metrizable spaces, $Z = \lim \mathbf{Z}$, and K a CW-complex. Suppose that for each $i \in \mathbb{N}$, $D \in \mathcal{B}(Z_i)$, and map $f : D \rightarrow K$, there exist $j \geq i$ and a map $g : Z_j \rightarrow K$ such that for all $x \in (g_i^j)^{-1}(D)$, $g(x) = f \circ g_i^j(x)$. Then $Z \tau K$. \square

We need to organize certain collections of homotopy classes. Here is the fundamental fact.

Lemma 2.11. *For each compact metrizable space X and compact polyhedron P , the set $[X, P]$ of homotopy classes of maps of X to P is countable.* \square

Definition 2.12. *For each compact metrizable space X and compact polyhedron P , select a countable collection $\mathcal{H}(X, P)$ consisting of one representative from each homotopy class in $[X, P]$.*

3. EXTENSOR LEMMA

For the remainder of this paper, I^∞ will denote the Hilbert cube, i.e., $I^\infty = \prod \{I_i \mid i \in \mathbb{N}\}$ where $I_i = I$ for each i . For each $j \in \mathbb{N}$, we factor I^∞ as $I^j \times I_j^\infty$. Let 0_j denote the element of I_j^∞ each of whose coordinates is 0. If $P \subset I^j$, then we may treat P as $P \times \{0_j\} \subset I^\infty$. The context should make this clear when we apply it. In this setting, an element of P becomes the element of I^∞ whose first j coordinates are the ones it inherits from P and whose remaining coordinates are all 0.

We shall use the metric ρ on I^∞ given by $\rho(x, y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i}$. For each $k \in \mathbb{N}$, $p_{k,\infty} : I^\infty \rightarrow I^k$ will denote the k -coordinate projection map, and if $j \leq k$, we will use $p_j^k : I^k \rightarrow I^j$ for the j -coordinate projection map. In case $x \in I^k$, then according to our convention $x = (x, 0, 0, \dots) \in I^\infty$, so one has that $p_j^k(x) = p_{j,\infty}(x) \in I^j$.

The main result of this section is Lemma 3.3. It provides us with a statement, see (3), about extending a map under the condition that a given compactum X has been embedded in I^∞ .

Lemma 3.1. *Let $X \subset I^\infty$ be compact and nonempty. Then there exist an increasing sequence (n_j) in \mathbb{N} with $n_1 = 1$, and a sequence (P_j) of compact polyhedra $P_j \subset I^{n_j}$, such that:*

- (1) *for all $j \in \mathbb{N}$, $X \subset \text{int}_{I^\infty}(P_j \times I_{n_j}^\infty) \subset N(X, \frac{2}{j})$, and*
- (2) *if $j \in \mathbb{N}_{\geq 2}$, then $p_{n_{j-1}}^{n_j}(P_j) \subset \text{int}_{I^{n_{j-1}}} P_{j-1}$.*

Proof. Put $n_1 = 1$ and $P_1 = I^1$. Then (1) is true in case $j = 1$ since $\text{diam } I^\infty \leq 1$, and (2) is true vacuously. Proceed by induction. Suppose that $j \in \mathbb{N}$, and we have found finite sequences $n_1 < \dots < n_j$ in \mathbb{N} and compact polyhedra P_1, \dots, P_j such that for $1 \leq s \leq j$, $P_s \subset I^{n_s}$, (1) is true up to j and (2) is true up to $j - 1$ in case $1 < j$.

One may choose $n_{j+1} \in \mathbb{N}$ such that $n_{j+1} > n_j$ and $X \subset p_{n_{j+1},\infty}(X) \times I_{n_{j+1}}^\infty \subset N(X, \frac{2}{j+1})$. There is a neighborhood V of $p_{n_{j+1},\infty}(X)$ such that $V \times I_{n_{j+1}}^\infty \subset N(X, \frac{2}{j+1})$. Choose a compact polyhedron P_{j+1} of $I^{n_{j+1}}$ so that, $p_{n_{j+1},\infty}(X) \subset \text{int}_{I^{n_{j+1}}} P_{j+1} \subset P_{j+1} \subset V$. This gives us (1) for $j + 1$.

Notice that (1) for j implies, $p_{n_j,\infty}(X) = p_{n_j}^{n_{j+1}} \circ p_{n_{j+1},\infty}(X) \subset \text{int}_{I^{n_j}} P_j$. Hence, $p_{n_{j+1},\infty}(X) \subset (p_{n_j}^{n_{j+1}})^{-1}(\text{int}_{I^{n_j}} P_j)$. Thus, making P_{j+1} smaller if necessary, we may have (1) and simultaneously, $P_{j+1} \subset (p_{n_j}^{n_{j+1}})^{-1}(\text{int}_{I^{n_j}} P_j)$. This achieves (2) for $j + 1$. \square

The condition (2) of Lemma 3.1 shows that if we replace j by $j + 1$, we find that $p_{n_j}^{n_{j+1}}(P_{j+1}) \subset \text{int}_{I^{n_j}} P_j$. This implies that $P_{j+1} \times I_{n_{j+1}}^\infty \subset$

$(\text{int}_{I^{n_j}} P_j) \times I_{n_j}^\infty$. This and (1) of Lemma 3.1 lead us to the next piece of information.

Corollary 3.2. *In the setting of Lemma 3.1,*

- (1) *for each $j \in \mathbb{N}$, $P_{j+1} \times I_{n_{j+1}}^\infty \subset \text{int}_{I^\infty}(P_j \times I_{n_j}^\infty)$, and*
- (2) $X = \bigcap \{P_j \times I_{n_j}^\infty \mid j \in \mathbb{N}\}$.

□

In reading (3) of the ensuing lemma, one should consult (2) of Lemma 3.1 to see that whenever $j \leq l$, then $p_{n_j}^{n_l}(P_l) \subset P_j$.

Lemma 3.3. *Let $X \subset I^\infty$, (n_j) , (P_j) be as in Lemma 3.1, and K be a CW-complex with $X \tau K$. Suppose that $j \in \mathbb{N}$ and B_j is a closed subset of P_j . For each $k \geq j$, let $B_k = (p_{n_j}^{n_k})^{-1}(B_j) \cap P_k$ and put $B_{j,\infty} = p_{n_j,\infty}^{-1}(B_j) \cap X$. The following are true.*

- (1) *For any open neighborhood S of $B_{j,\infty}$ in I^∞ , there exists $k \geq j$ such that for all $l \geq k$, $B_l \subset S$.¹*
- (2) *If $f : B_j \rightarrow K$ is a map, then there exists $k \geq j$ such that for all $l \geq k$, there is a map $f^* : P_l \rightarrow K$ that extends the composition $f \circ p_{n_j}^{n_l}|_{B_l} : B_l \rightarrow K$ where we treat $p_{n_j}^{n_l}|_{B_l} : B_l \rightarrow B_j$.*
- (3) *Suppose that $j \in \mathbb{N}$, T_j is a triangulation of P_j , L is a subcomplex of T_j , and that $|N_{|L|,T_j}|$, $N_{|L|,T_j}$ being the simplicial neighborhood of $|L|$ in T_j , is a regular neighborhood of $|L|$ in $|T_j|$. Assume that $k \geq j$ is as in (2) with $B_j = |L|$, $l \geq k$, and $g : P_l \rightarrow |T_j|$ is a map which is a T_j -modification of $p_{n_j}^{n_l}|_{P_l} : P_l \rightarrow P_j = |T_j|$. Let $f : |L| \rightarrow K$ be a map and $E = g^{-1}(|L|) \subset P_l$. Then there is a map $g^* : P_l \rightarrow K$ that extends the composition $f \circ g|_E : E \rightarrow K$.*

Proof. Let S be an open neighborhood of $B_{j,\infty}$ in I^∞ . If the conclusion of (1) is not true, then there is an increasing sequence (m_i) in \mathbb{N} , $m_1 \geq j$, so that for each i , there exists $b_i \in B_{m_i} \setminus S \subset P_{m_i}$. Passing to a subsequence if necessary, we may assume that the sequence (b_i) in the compactum $I^\infty \setminus S$ converges in I^∞ to $b \in I^\infty \setminus S$. Applying Corollary 3.2(1,2) along with the fact that $b_i \in P_{m_i}$, one sees that $b \in X \setminus B_{j,\infty}$, from which we deduce that $p_{n_j,\infty}(b) \notin B_j$.

For each i , $p_{n_j}^{s_i}(b_i) = p_{n_j,\infty}(b_i) \in B_j$, $s_i = n_{m_i}$. Hence $\{p_{n_j,\infty}(b_i) \mid i \in \mathbb{N}\} \subset B_j$. Since B_j is closed in P_j , $p_{n_j,\infty}$ is a map, and (b_i) converges to b , then $p_{n_j,\infty}(b) \in B_j$, a contradiction. This yields (1). Now to prove (2).

Employing Lemma 2.5, select an open cover \mathcal{V} of K such that for any space Y , any maps $g_1 : Y \rightarrow K$ and $g_2 : Y \rightarrow K$ that are \mathcal{V} -close are homotopic. Let \mathcal{V}_1 be an open cover of K that star-refines \mathcal{V} . Choose an open cover \mathcal{W} of B_j such that if $W \in \mathcal{W}$, then there exists $V_W \in \mathcal{V}_1$ with $f(W) \subset V_W$.

Observe that $B_{j,\infty}$ is a closed subset of X and that $p_{n_j,\infty}(B_{j,\infty}) \subset B_j$. Since $X \tau K$, then the map $f \circ p_{n_j,\infty}|_{B_{j,\infty}} : B_{j,\infty} \rightarrow K$ extends to a map $h : U \rightarrow K$ where U is an open neighborhood of X in I^∞ . Select an open cover \mathcal{R} of X in U having the property that for each $R \in \mathcal{R}$, there exists $V_R \in \mathcal{V}_1$ with $h(R) \subset V_R$. Let $S = \bigcup \mathcal{R} \subset U$. Then S is an open neighborhood of X in I^∞ . So by (1), we may choose $k \in \mathbb{N}$ so that for all $l \geq k$, $B_l \subset S$. Using Corollary 3.2(1,2), we may also require that for such l , $P_l = P_l \times \{0_{n_l}\} \subset S$.

¹Statement (1) is true independently of the condition $X \tau K$.

Put $B^* = B_{j,\infty} \cup \bigcup \{B_l \mid l \geq k\} \subset S$. Then of course $p_{j,\infty} : B^* \rightarrow B_j$ is a map. For each $b \in B_{j,\infty}$, select a neighborhood E_b of b in B^* such that $p_{j,\infty}(E_b)$ is contained in an element W_b of \mathcal{W} and that in addition, there exists $R_b \in \mathcal{R}$ with $E_b \subset R_b$. Let $S_0 = \bigcup \{E_b \mid b \in B_{j,\infty}\}$. Then S_0 is an open neighborhood of $B_{j,\infty}$ in $B^* \subset I^\infty$. So there is an open subset S_1 of I^∞ having the property that $S_1 \cap B^* = S_0$. Plainly, S_1 is an open neighborhood of $B_{j,\infty}$ in I^∞ . An application of (1) with S_1 in place of S gives us the existence of a $k_1 \geq k$ so that for all $l \geq k_1$, we have $B_l \subset S_1$. But then $B_l \subset S_1 \cap B^* = S_0$.

We are going to show that $f \circ p_{n_j}^{n_l} | B_l : B_l \rightarrow K$ is homotopic to $h | B_l : B_l \rightarrow K$. For in that case, if we define $h_0 = h | B_l : B_l \rightarrow K$, then of course since $P_l \subset S$, h_0 extends to the map $h | P_l : P_l \rightarrow K$, and the homotopy extension theorem will complete our proof.

Let $x \in B_l$. It will be sufficient to show that $f \circ p_{n_j}^{n_l}(x)$ and $h_0(x)$ lie in an element of \mathcal{V} . There exists $b \in B_{j,\infty}$ such that $x \in E_b$. Now $b \in E_b \subset R_b \in \mathcal{R}$. It follows that there is an element $V_1 \in \mathcal{V}_1$ with $\{h(b), h(x)\} = \{h(b), h_0(x)\} \subset V_1$. One sees from the definition of h and the fact that $b \in B_{j,\infty}$, that $h(b) = f \circ p_{j,\infty}(b)$. So we have that $\{f \circ p_{j,\infty}(b), h_0(x)\} \subset V_1 \in \mathcal{V}_1$. We know that $p_{j,\infty}(b) \in p_{j,\infty}(E_b) \subset W_b \in \mathcal{W}$. Thus $f \circ p_{j,\infty}(b) \in f \circ p_{j,\infty}(E_b) \subset f(W_b) \subset V_2$ for some $V_2 \in \mathcal{V}_1$. Now $x \in E_b \cap B_l \subset E_b \cap P_l \subset E_b \cap I^{n_l}$, so $p_{j,\infty}(x) = p_{n_j}^{n_l}(x) \in p_{j,\infty}(E_b)$, and we see that $f \circ p_{n_j}^{n_l}(x) \in f \circ p_{j,\infty}(E_b) \subset V_2$. Hence, $\{f \circ p_{j,\infty}(b), f \circ p_{n_j}^{n_l}(x)\} \subset V_2$. Since \mathcal{V}_1 is a star-refinement of \mathcal{V} , $f \circ p_{j,\infty}(b) \in V_1 \cap V_2$, $h_0(x) \in V_1$, and $f \circ p_{n_j}^{n_l}(x) \in V_2$, one may find $V \in \mathcal{V}$ with $\{f \circ p_{n_j}^{n_l}(x), h_0(x)\} \subset V_1 \cup V_2 \subset V$.

Lastly, we prove (3). Since $|N_{|L|,T_j}|$ is a regular neighborhood of $|L|$ in $|T_j|$ and hence $|L|$ is a retract of this regular neighborhood, then there is no loss of generality in assuming that $f : |N_{|L|,T_j}| \rightarrow K$. Now we apply (2) with $B_j = |N_{|L|,T_j}|$. So there exists $k \geq j$ such that for all $l \geq k$, there is a map $f^* : P_l \rightarrow K$ that extends the composition $f \circ p_{n_j}^{n_l} | B_l : B_l \rightarrow K$ where

$$B_l = (p_{n_j}^{n_l})^{-1}(B_j) \cap P_l = (p_{n_j}^{n_l})^{-1}(|N_{|L|,T_j}|) \cap P_l.$$

Here we treat $p_{n_j}^{n_l} | B_l : B_l \rightarrow B_j = |N_{|L|,T_j}|$. By definition (see(3)),

$$E = g^{-1}(|L|) \subset P_l.$$

Let us demonstrate that,

$$(\dagger_1) \ p_{n_j}^{n_l}(E) \subset |N_{|L|,T_j}|.$$

Suppose, for the sake of contradiction, that $x \in E$ and $p_{n_j}^{n_l}(x) \in P_j \setminus |N_{|L|,T_j}| = |T_j| \setminus |N_{|L|,T_j}|$. There is a simplex $\sigma \in T_j$ with $p_{n_j}^{n_l}(x) \in \text{int}(\sigma)$ and so that $\sigma \cap |L| = \emptyset$. Applying the fact that g is a T_j -modification of $p_{n_j}^{n_l} : P_l \rightarrow P_j$, one sees that $g(x) \in \sigma$. But then $g(x) \notin |L|$, which is false. So we get (\dagger_1) .

The preceding and Lemma 2.2 show that the maps $g | E : E \rightarrow |L| \subset |N_{|L|,T_j}|$ and $p_{n_j}^{n_l} | E : E \rightarrow |N_{|L|,T_j}|$ are homotopic in $|N_{|L|,T_j}|$. Hence the compositions $f \circ g | E : E \rightarrow K$ and $f \circ p_{n_j}^{n_l} | E : E \rightarrow K$ are homotopic.

Using (\dagger_1) , one has,

$$(\dagger_2) \ E = g^{-1}(|L|) \cap P_l \subset (p_{n_j}^{n_l})^{-1}(|N_{|L|,T_j}|) \cap P_l = (p_{n_j}^{n_l})^{-1}(B_j) \cap P_l = B_l.$$

The map $f^* : P_l \rightarrow K$ extends the composition $f \circ p_{n_j}^{n_l} : B_l \rightarrow K$. Using this and (\dagger_2) , one sees that f^* extends the composition $f \circ p_{n_j}^{n_l} | E : E \rightarrow K$.

So an application of the homotopy extension theorem gives us the desired map g^* , completing our proof of (3). \square

4. EXTENSION DIMENSION

In order to strengthen forthcoming results, we are going to employ the notion of extension dimension. The reader can find a good exposition of this in [IR], but we will provide all the necessary ideas in this section.

Let \mathcal{C} be a class of spaces and \mathcal{T} a class of CW-complexes. For each $K \in \mathcal{T}$, there is a subclass $[K]_{(\mathcal{C}, \mathcal{T})} \subset \mathcal{T}$ which is called the *extension type* of K relative to $(\mathcal{C}, \mathcal{T})$. Indeed, the extension types form a decomposition of \mathcal{T} . There is a partial order $\leq_{(\mathcal{C}, \mathcal{T})}$ on the class of extension types (see page 384 of [IR]). We are not going to explain it here, but we shall indicate later how this comes into play for us.

Henceforward, \mathcal{C} will be the class of metrizable compacta and \mathcal{T} will be the class of CW-complexes. We only need to mention that metrizable spaces are stratifiable ((SP7) on p. 386 of [IR]), and hence the results of [IR] will apply to our choice of \mathcal{C} .

Proposition 4.1. *For all $X \in \mathcal{C}$ and $K \in \mathcal{T}$, $X\tau K$ if and only if $X\tau L$ for all $L \in [K]_{(\mathcal{C}, \mathcal{T})}$.*

On the basis of Proposition 4.1, if P is an extension type, then one usually writes $X\tau P$ to mean that $X\tau K$ for all $K \in P$. Fix $X \in \mathcal{C}$ and consider the class $\mathcal{E}(X)$ consisting of those extension types P relative to $(\mathcal{C}, \mathcal{T})$ with $X\tau P$. If $(\mathcal{E}(X), \leq_{(\mathcal{C}, \mathcal{T})})$ has an initial element², then that element is called the *extension dimension* of X with respect to $(\mathcal{C}, \mathcal{T})$, written $\text{extdim}_{(\mathcal{C}, \mathcal{T})} X$. Here are the facts that are salient to us.

By Corollary 5.4 of [IR],

Proposition 4.2. *The extension dimension, $\text{extdim}_{(\mathcal{C}, \mathcal{T})} X$, exists for every metrizable compactum X .*

Proposition 4.3. *For all $K, L \in \mathcal{T}$, $[K]_{(\mathcal{C}, \mathcal{T})} = [L]_{(\mathcal{C}, \mathcal{T})}$ whenever $K \simeq L$.*

Proposition 4.4. *For every metrizable compactum X , there exists a polyhedron K such that $[K]_{(\mathcal{C}, \mathcal{T})} = \text{extdim}_{(\mathcal{C}, \mathcal{T})} X$.*

Lemma 4.5. *Let X be a metrizable compactum and K a CW-complex with $[K]_{(\mathcal{C}, \mathcal{T})} = \text{extdim}_{(\mathcal{C}, \mathcal{T})} X$. Then for every metrizable compactum Y , $Y\tau K$ implies that $Y\tau L$ whenever L is a CW-complex and $X\tau L$. \square*

5. SIMPLICIAL RESOLUTION

Often when a space is given as the limit of an inverse system, that inverse system is called a “resolution” of the space. It has been shown by S. Mardešić (see [Ma]) that in general it is impossible to resolve a metrizable compactum by an inverse sequence of triangulated compact polyhedra in which each bonding map is simplicial with respect to these triangulations. But having such a resolution might be valuable in extension theory. Starting with a nonempty metrizable compactum X we are going to prove the existence of a certain “simplicial” inverse sequence (see Definition 5.3). This will be an

²An initial element $s_0 \in S$ of a partially ordered set (S, \leq) is understood in the following sense: for every $s \in S$, $s_0 \leq s$. Such s_0 , if it exists, is unique.

important step in reaching our goal of finding a “replacement” of the original space X for the purpose of extension theory. The construction that yields what we want is found in Lemma 5.2.

Definition 5.1. *For a given nonempty simplicial complex K , let $\mathcal{F}(K)$ be a countable set of finite simplicial complexes having the property that for each $N \in \mathcal{F}(K)$ there exists a subcomplex N^* of K that is simplicially isomorphic to N and such that for each finite subcomplex E of K , there exists an element N of $\mathcal{F}(K)$ that is simplicially isomorphic to E .*

Lemma 5.2. *Let $X \subset I^\infty$ be compact and nonempty and K be a simplicial complex such that $[[K]]_{(\mathcal{C}, \mathcal{T})} = \text{extdim}_{(\mathcal{C}, \mathcal{T})} X$. Select a bijective function $\eta : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, denote*

$$(*_1) \quad \eta(j) = (s_j, t_j) \text{ for each } j \in \mathbb{N},$$

and require that,

$$(*_2) \quad s_j \leq j \text{ for each } j \in \mathbb{N}.$$

Then there exists a sequence (\mathcal{S}_j) , $\mathcal{S}_j = (n_j, P_j, \epsilon_j, \tilde{T}_j, f_j^{j+1}, \delta_j, T_j, \varphi_j, g_j^{j+1})$, such that (n_j) is an increasing sequence in \mathbb{N} , (P_j) is a sequence of compact polyhedra, for each j , $P_j \subset I^{n_j}$, $p_{n_j}^{n_{j+1}}(P_{j+1}) \subset P_j$, $f_j^{j+1} : P_{j+1} \rightarrow P_j$ is a map, (ϵ_j) and (δ_j) are sequences of positive real numbers, and \tilde{T}_j and T_j are triangulations of P_j , T_j being a subdivision of \tilde{T}_j . The sequence will be constructed so that $f_j^{j+1} : |\tilde{T}_{j+1}| \rightarrow |T_j|$ is a simplicial approximation of $p_{n_j}^{n_{j+1}}|P_{j+1} : P_{j+1} \rightarrow P_j$, $\varphi_j : |T_j| \rightarrow |\tilde{T}_j|$ is a simplicial approximation of the identity map of $P_j = |T_j|$, and we shall define $g_j^{j+1} = f_j^{j+1} \circ \varphi_{j+1} : |T_{j+1}| \rightarrow |T_j|$. For each $j \in \mathbb{N}$, we shall index the countable set $\mathfrak{F}_j = \bigcup \{\mathcal{H}(D, |N|) \mid (D, N) \in \mathcal{B}(P_j) \times \mathcal{F}(K)\}$ as $\{f_{j,k} \mid k \in \mathbb{N}\}$ where $f_{j,k} : D_{j,k} \rightarrow |N_{j,k}|$. All eight of the following conditions will be satisfied simultaneously by these choices:

- (1) *if $j > 1$, $u, v \in I^\infty$ and $\rho(u, v) < \epsilon_j$, then $\rho(p_{n_s, \infty}(u), p_{n_s, \infty}(v)) < \delta_s$ for all $1 \leq s < j$,*
- (2) $5 \cdot 2^{-n_j} < \epsilon_j$,
- (3) $\delta_j < 2^{1-n_j}$,
- (4) $\text{mesh } T_j < \frac{\delta_j}{2}$,
- (5) *the map $f_{s_j, t_j} \circ g_{s_j}^j : (g_{s_j}^j)^{-1}(D_{s_j, t_j}) : (g_{s_j}^j)^{-1}(D_{s_j, t_j}) \rightarrow |N_{s_j, t_j}|$ extends to a map $\hat{f}_{s_j, t_j} : |E_{s_j, t_j}| \rightarrow |N_{s_j, t_j}|$, where E_{s_j, t_j} is the simplicial neighborhood of $(g_{s_j}^j)^{-1}(D_{s_j, t_j})$ in T_j ,*
- (6) $|E_{s_j, t_j}^*|$ *is a regular neighborhood of $|E_{s_j, t_j}|$ in T_j where E_{s_j, t_j}^* is the simplicial neighborhood of $|E_{s_j, t_j}|$ in T_j ,*
- (7) *for each $x \in X$, there exists $v_{x,j} \in \tilde{T}_j^{(0)}$ such that $\overline{N}(p_{n_j, \infty}(x), 2\delta_j) \cap P_j \subset \overline{\text{st}}(v_{x,j}, \tilde{T}_j) \subset \overline{N}(p_{n_j, \infty}(x), \epsilon_j) \cap P_j$, and*
- (8) *whenever $j > 1$ and $1 \leq s < j$, the simplicial map $g_s^j : |T_j| \rightarrow |T_s|$ is a simplicial approximation of the restricted projection $p_{n_s}^{n_j}|P_j : P_j \rightarrow P_s$.*

Proof. Let (n_j) be an increasing sequence in \mathbb{N} as in Lemma 3.1. There is also a sequence (P_j) of polyhedra given there, and for each of these we select the set \mathfrak{F}_j indexed as in the hypothesis. In the ensuing proof we might have to find an increasing sequence (j_i) in \mathbb{N} and replace (n_j) with the nondecreasing

subsequence (m_i) , $m_i = n_{j_i}$. To conserve notation, we shall not introduce the symbols m_i , but rather will simply rename n_j as needed and rely on Lemma 3.1 and Corollary 3.2 to help fill in any gaps that this might seem to present. A similar case will also apply to the polyhedra P_j . Note that $n_1 = 1$.

Choose $\epsilon_1 = 5$ and a triangulation \tilde{T}_1 of $P_1 = I^1$ with $\text{mesh } \tilde{T}_1 < \frac{5}{2} = \frac{\epsilon_1}{2}$. Let λ_1 be a Lebesgue number of the open cover $\mathcal{U}_1 = \{\text{st}(v, \tilde{T}_1) \mid v \in \tilde{T}_1^{(0)}\}$ of $P_1 = I^1$, and pick $0 < \delta_1 = \min\{\frac{\lambda_1}{3}, 2^{-1}\}$.

Using $(*)_1$, $(*)_2$, observe that $\eta(1) = (s_1, t_1) = (1, t_1)$ for some $t_1 \in \mathbb{N}$. The map $f_{1,t_1} = f_{s_1,t_1} : D_{s_1,t_1} \rightarrow |N_{s_1,t_1}|$ lies in $\mathfrak{F}_{s_1} = \mathfrak{F}_1$, so its domain D_{s_1,t_1} is a closed subset of P_1 and its range is a compact polyhedron. Hence f_{s_1,t_1} extends to a map $f_{s_1,t_1}^* : D_{s_1,t_1}^* \rightarrow |N_{s_1,t_1}|$, where D_{s_1,t_1}^* is a neighborhood of D_{s_1,t_1} in P_1 . Select a triangulation T_1 of P_1 that refines \tilde{T}_1 and so that $\text{mesh } T_1 < \frac{\delta_1}{2}$. We may also assume about the triangulation T_1 that $|E_{s_1,t_1}| \subset D_{s_1,t_1}^*$, where E_{s_1,t_1} is the simplicial neighborhood of D_{s_1,t_1} with respect to T_1 . Employing f_{s_1,t_1}^* and the preceding, we see that f_{s_1,t_1} extends to a map $\hat{f}_{s_1,t_1} : |E_{s_1,t_1}| \rightarrow |N_{s_1,t_1}|$. We may go even further and require that $|E_{s_1,t_1}^*|$ is a regular neighborhood of $|E_{s_1,t_1}|$ in $|T_1|$, where E_{s_1,t_1}^* is the simplicial neighborhood of $|E_{s_1,t_1}|$ in T_1 . Let $\varphi_1 : |T_1| \rightarrow |\tilde{T}_1|$ be a simplicial approximation of the identity map of P_1 .

Since $2\delta_1 < \lambda_1$, then for each $x \in X$ we may choose $v_{x,1} \in \tilde{T}_1^{(0)}$ such that $\overline{N}(p_{n_1,\infty}(x), 2\delta_1) \subset \text{st}(v_{x,1}, \tilde{T}_1)$. For a given $y \in \overline{\text{st}}(v_{x,1}, \tilde{T}_1)$, $\rho(y, p_{n_1,\infty}(x)) \leq 2 \text{mesh } \tilde{T}_1 < \epsilon_1$, so $\overline{\text{st}}(v_{x,1}, \tilde{T}_1) \subset \overline{N}(p_{n_1,\infty}(x), \epsilon_1) = \overline{N}(p_{n_1,\infty}(x), \epsilon_1) \cap P_1$. It follows that, $\overline{N}(p_{n_1,\infty}(x), 2\delta_1) = \overline{N}(p_{n_1,\infty}(x), 2\delta_1) \cap P_1 \subset \overline{\text{st}}(v_{x,1}, \tilde{T}_1) \subset \overline{N}(p_{n_1,\infty}(x), \epsilon_1) \cap P_1$.

Taking the preceding as the first step in a recursion, then all of (1)–(8) hold true for $j = 1$. Now assume that $i \in \mathbb{N}$, and that we have completed the construction of \mathcal{S}_j for each $1 \leq j \leq i$ in accordance with (1)–(8). Aside from the inductive assumptions, and as mentioned above, we insist on one proviso. We shall agree that the numbers $\{n_1, \dots, n_i\}$ are in reality $\{n_{j_1}, \dots, n_{j_i}\}$ where $j_1 < \dots < j_i$, so we get a finite nondecreasing subsequence of the given infinite sequence (n_j) (retaining the symbol n_j in order to conserve notation). Now we proceed for the $(i+1)$ -step.

Applying the uniform continuity of the coordinate projections of I^∞ , select $0 < \epsilon_{i+1}$ so that if $u, v \in I^\infty$ and $\rho(u, v) < \epsilon_{i+1}$, then for each $1 \leq s \leq i$, $\rho(p_{n_s,\infty}(u), p_{n_s,\infty}(v)) < \delta_s$. This achieves (1) for $j = i+1$.

Choose $n_{i+1} > n_i$ so that (2) is satisfied for $j = i+1$. As a consequence of Lemma 3.1(2), $p_{n_s}^{n_j}(P_j) \subset P_s$ for all $1 \leq s < j \leq i+1$. So we can write the restrictions of the projections as $p_{n_s}^{n_j}|P_j : P_j \rightarrow P_s$, for such s . Keep in mind that $P_{i+1} \subset I^{n_{i+1}}$. Select a triangulation \tilde{T}_{i+1} of P_{i+1} with $\text{mesh } \tilde{T}_{i+1} < \frac{\epsilon_{i+1}}{2}$ and so that at the same time we may find a simplicial approximation $f_i^{i+1} : |\tilde{T}_{i+1}| \rightarrow |T_i|$ to the map $p_{n_i}^{n_{i+1}}|P_{i+1} : P_{i+1} \rightarrow P_i$. Let λ_{i+1} be a Lebesgue number of the open cover $\mathcal{U}_{i+1} = \{\text{st}(v, \tilde{T}_{i+1}) \mid v \in \tilde{T}_{i+1}^{(0)}\}$ of P_{i+1} , and pick $0 < \delta_{i+1} = \min\{\frac{\lambda_{i+1}}{3}, 2^{1-n_{i+1}}\}$. This gives us (3) for $j = i+1$.

Using $(*)_1$, $(*)_2$, observe that $\eta(i+1) = (s_{i+1}, t_{i+1})$ for some $t_{i+1} \in \mathbb{N}$ and where $s_{i+1} \leq i+1$. The map $f_{s_{i+1},t_{i+1}} : D_{s_{i+1},t_{i+1}} \rightarrow |N_{s_{i+1},t_{i+1}}|$ lies

in $\mathfrak{F}_{s_{i+1}}$, so its domain $D_{s_{i+1}, t_{i+1}}$ is a closed subset of $P_{s_{i+1}}$ and its range is a compact polyhedron. So the map $f_{s_{i+1}, t_{i+1}} \circ g_{s_{i+1}}^{i+1} | (g_{s_{i+1}}^{i+1})^{-1}(D_{s_{i+1}, t_{i+1}}) : (g_{s_{i+1}}^{i+1})^{-1}(D_{s_{i+1}, t_{i+1}}) \rightarrow |N_{s_{i+1}, t_{i+1}}|$ extends to a map $f_{s_{i+1}, t_{i+1}}^* : D_{s_{i+1}, t_{i+1}}^* \rightarrow |N_{s_{i+1}, t_{i+1}}|$, where $D_{s_{i+1}, t_{i+1}}^*$ is a neighborhood of $(g_{s_{i+1}}^{i+1})^{-1}(D_{s_{i+1}, t_{i+1}})$ in P_{i+1} . Select a triangulation T_{i+1} of P_{i+1} that refines \tilde{T}_{i+1} and so that $\text{mesh } T_{i+1} < \frac{\delta_{i+1}}{2}$; this accomplishes (4) for $j = i + 1$. We may also assume about the triangulation T_{i+1} that $|E_{s_{i+1}, t_{i+1}}| \subset D_{s_{i+1}, t_{i+1}}^*$, where $E_{s_{i+1}, t_{i+1}}$ is the simplicial neighborhood of $(g_{s_{i+1}}^{i+1})^{-1}(D_{s_{i+1}, t_{i+1}})$ with respect to T_{i+1} . Employing $f_{s_{i+1}, t_{i+1}}^*$ and the preceding, we see that $f_{s_{i+1}, t_{i+1}}$ extends to a map $\hat{f}_{s_{i+1}, t_{i+1}} : |E_{s_{i+1}, t_{i+1}}| \rightarrow |N_{s_{i+1}, t_{i+1}}|$, so we get (5). We may go even further and require that $|E_{s_{i+1}, t_{i+1}}^*|$ is a regular neighborhood of $|E_{s_{i+1}, t_{i+1}}|$ in $|T_{i+1}|$, where $E_{s_{i+1}, t_{i+1}}^*$ is the simplicial neighborhood of $|E_{s_{i+1}, t_{i+1}}|$ in T_{i+1} . This yields (6). Let $\varphi_{i+1} : |T_{i+1}| \rightarrow |\tilde{T}_{i+1}|$ be a simplicial approximation of the identity map of P_{i+1} .

Since $2\delta_{i+1} < \lambda_{i+1}$, then for each $x \in X$ we may choose $v_{x, i+1} \in \tilde{T}_{i+1}^{(0)}$ such that $\overline{N}(p_{n_{i+1}, \infty}(x), 2\delta_{i+1}) \cap P_{i+1} \subset \text{st}(v_{x, i+1}, \tilde{T}_{i+1})$. For a given $y \in \overline{\text{st}}(v_{x, i+1}, \tilde{T}_{i+1})$, $\rho(y, p_{n_{i+1}, \infty}(x)) \leq 2 \text{mesh } \tilde{T}_{i+1} < \epsilon_{i+1}$, so $\overline{\text{st}}(v_{x, i+1}, \tilde{T}_{i+1}) \subset \overline{N}(p_{n_{i+1}, \infty}(x), \epsilon_{i+1}) \cap P_{i+1}$. It follows that, $\overline{N}(p_{n_{i+1}, \infty}(x), 2\delta_{i+1}) \cap P_{i+1} \subset \overline{\text{st}}(v_{x, i+1}, \tilde{T}_{i+1}) \subset \overline{N}(p_{n_{i+1}, \infty}(x), \epsilon_{i+1}) \cap P_{i+1}$. We have achieved (7) for $j = i + 1$. Put $g_i^{i+1} = f_i^{i+1} \circ \varphi_{i+1} : |T_{i+1}| \rightarrow |T_i|$. We see that g_i^{i+1} is simplicial. It follows that for each $1 \leq s < i + 1$, $g_s^{i+1} : |T_{i+1}| \rightarrow |T_s|$ is a simplicial map.

The next thing to establish is (8) for $j = i + 1$. We have to show that if $1 \leq s < i + 1$, then $g_s^{i+1} : |T_{i+1}| \rightarrow |T_s|$ is a simplicial approximation of $p_{n_s}^{n_{i+1}} | P_{i+1} : P_{i+1} \rightarrow P_s$. The inductive assumption is that (8) is true whenever $1 \leq i_0 < i$ and $j = i_0 + 1$. This means that if $1 \leq s < i_0 + 1$, then $g_s^{i_0+1} : |T_{i_0+1}| \rightarrow |T_s|$ is a simplicial approximation of $p_{n_s}^{n_{i_0+1}} | P_{i_0+1} : P_{i_0+1} \rightarrow P_s$. Let us determine a fact that will be useful twice.

(†) Let $x \in P_{i+1}$. There are a unique $\sigma \in T_i$ with $p_{n_i}^{n_{i+1}}(x) \in \text{int}(\sigma)$, $\tau \in T_{i+1}$ with $x \in \text{int}(\tau)$, and $\tau^* \in \tilde{T}_{i+1}$ with $x \in \text{int}(\tau^*)$. Thus, $\tau \subset \tau^*$, and $\varphi_{i+1}(x) \in \tau^*$. Since $f_i^{i+1} : |\tilde{T}_{i+1}| \rightarrow |T_i|$ is a simplicial approximation of $p_{n_i}^{n_{i+1}} | P_{i+1} : P_{i+1} \rightarrow P_i$, $\sigma \in T_i$, $x \in \text{int}(\tau^*)$, and $p_{n_i}^{n_{i+1}}(x) \in \text{int}(\sigma)$, then $f_i^{i+1}(\tau^*) \subset \sigma$.

Consider first the case that $s = i$. We have to show that the simplicial map $g_i^{i+1} : |T_{i+1}| \rightarrow |T_i|$ is a simplicial approximation of $p_{n_i}^{n_{i+1}} | P_{i+1} : P_{i+1} \rightarrow P_i$. The result of (†) shows that $g_i^{i+1}(x) = f_i^{i+1} \circ \varphi_{i+1}(x) \in f_i^{i+1}(\tau^*) \subset \sigma$, as required for (8).

The other case is that $s < i$. The inductive assumption gives us:

(*) $g_s^i : |T_i| \rightarrow |T_s|$ is a simplicial approximation of $p_{n_s}^{n_i} | P_i : P_i \rightarrow P_s$.

Choose $\kappa \in T_s$ such that $p_{n_s}^{n_{i+1}}(x) = p_{n_s}^{n_i} \circ p_{n_i}^{n_{i+1}}(x) \in \text{int } \kappa$. By (*), $g_s^i \circ p_{n_i}^{n_{i+1}}(x) \in \kappa$. Take σ from (†). Then $p_{n_i}^{n_{i+1}}(x) \in \text{int}(\sigma)$, $\sigma \in T_i$, and $g_s^i \circ p_{n_i}^{n_{i+1}}(x) \in \kappa$, so since g_s^i is simplicial, one has that $g_s^i(\sigma) \subset \kappa$. In the case $s = i$, we showed that $g_i^{i+1}(x) \in \sigma$. It follows that $g_s^{i+1}(x) = g_s^i \circ g_i^{i+1}(x) \in g_s^i(\sigma) \subset \kappa$, which is what we need to complete the proof of (8). \square

Definition 5.3. For each nonempty metrizable compactum X , select an embedding $X \hookrightarrow I^\infty$, let K be a simplicial complex such that $[[K]]_{(C, \mathcal{T})} = \text{extdim}_{(C, \mathcal{T})} X$, and choose a sequence (S_j) as in Lemma 5.2. Then $\mathbf{Z} =$

$(|T_j|, g_j^{j+1})$ will be called an **induced inverse sequence** for X . We shall usually use the term *induced inverse sequence* without reference to the sequence (S_j) , but the latter will always be available if needed in an argument.

6. EXTENSION-THEORETIC PROPERTY OF AN INDUCED INVERSE SEQUENCE

Theorem 6.1 provides the second step in showing how to replace a given nonempty metrizable compactum with a better one for the purposes of extension theory.

Theorem 6.1. *Let X be a nonempty metrizable compactum and suppose that $\mathbf{Z} = (|T_j|, g_j^{j+1})$ is an induced inverse sequence for X as in Definition 5.3. Put $Z = \lim \mathbf{Z}$. If K_0 is a CW-complex and $X\tau K_0$, then $Z\tau K_0$.*

Proof. Let us incorporate all the notation from Lemma 5.2. It follows from Theorem 4.1, that we only have to show that $Z\tau|K|$. Let $j \in \mathbb{N}$, $D \in \mathcal{B}(P_j)$ ($P_j = |T_j|$), and $f : D \rightarrow |K|$ a map. We are going to find $k \geq j$ and a map $g : P_k \rightarrow |K|$ such that for all $x \in (g_j^k)^{-1}(D)$, $g(x) = f \circ g_j^k(x)$. According to Proposition 2.10, that will complete our proof.

There is a finite subcomplex F of K with $f(D) \subset |F|$. Choose an element N of $\mathcal{F}(K)$ that is simplicially isomorphic to F and let $\phi : |F| \rightarrow |N|$ and $\psi : |N| \rightarrow |F|$ be inverse homeomorphisms. Put $f_0 = f : D \rightarrow |F|$, and then $f^* = \phi \circ f_0 : D \rightarrow |N|$. There exists $l \in \mathbb{N}$ so that $N_{j,l} = N$, $D_{j,l} = D$, and $f_{j,l} \simeq f^*$, where $f_{j,l} : D_{j,l} \rightarrow |N_{j,l}|$. (See Definition 5.1 and \mathfrak{F}_j of Lemma 5.2).

Since η of Lemma 5.2 is surjective, choose j^* so that $\eta(j^*) = (j, l)$. This means that $(j, l) = (s_{j^*}, t_{j^*})$ and, of course, $s_{j^*} = j \leq j^*$ (see $(*)_1$ and $(*)_2$ of Lemma 5.2). By this, one can see that $D = D_{s_{j^*}, t_{j^*}}$, $N_{j,l} = N_{s_{j^*}, t_{j^*}}$, $f_{j,l} = f_{s_{j^*}, t_{j^*}} \simeq f^* : D_{s_{j^*}, t_{j^*}} \rightarrow |N_{s_{j^*}, t_{j^*}}|$, and that $\psi : |N_{s_{j^*}, t_{j^*}}| \rightarrow |F|$. With j^* in place of j , employ (5) and (6) of Lemma 5.2. The map $f_{s_{j^*}, t_{j^*}} \circ g_{s_{j^*}}^{j^*} |(g_{s_{j^*}}^{j^*})^{-1}(D_{s_{j^*}, t_{j^*}}) : (g_{s_{j^*}}^{j^*})^{-1}(D_{s_{j^*}, t_{j^*}}) \rightarrow |N_{s_{j^*}, t_{j^*}}|$ extends to a map $\hat{f}_{s_{j^*}, t_{j^*}} : |E_{s_{j^*}, t_{j^*}}| \rightarrow |N_{s_{j^*}, t_{j^*}}|$ where $E_{s_{j^*}, t_{j^*}}$ is the simplicial neighborhood of $(g_{s_{j^*}}^{j^*})^{-1}(D_{s_{j^*}, t_{j^*}})$ in T_{j^*} , and $|E_{s_{j^*}, t_{j^*}}^*|$ is a regular neighborhood of $|E_{s_{j^*}, t_{j^*}}|$ in $|T_{j^*}|$, where $E_{s_{j^*}, t_{j^*}}^*$ is the simplicial neighborhood of $|E_{s_{j^*}, t_{j^*}}|$ in T_{j^*} . Now put $\hat{f} = \psi \circ \hat{f}_{s_{j^*}, t_{j^*}} : |E_{s_{j^*}, t_{j^*}}| \rightarrow |F| \subset |K|_{\text{CW}}$.

Next we put Lemma 3.3 into play. Replace j by j^* and L by $E_{s_{j^*}, t_{j^*}}$ in (3) of that Lemma, and make the observation that the simplicial neighborhood $E_{s_{j^*}, t_{j^*}}^*$ of $|E_{s_{j^*}, t_{j^*}}|$ in T_{j^*} produces the desired regular neighborhood $|E_{s_{j^*}, t_{j^*}}^*|$ of $|E_{s_{j^*}, t_{j^*}}|$ in T_{j^*} . Assume that $k \geq j^*$ is as in (2) of Lemma 3.3 with $B_{j^*} = |E_{s_{j^*}, t_{j^*}}|$ playing the role of B_j , and use $g_{j^*}^k : P_k \rightarrow |T_{j^*}|$ as the map g which is a T_{j^*} -modification of $p_{n_{j^*}}^{n_k} |P_k : P_k \rightarrow P_{j^*} = |T_{j^*}|$ (in this instance we just take $l = k$ for the l of Lemma 3.3). The map f of (3) of the cited lemma will be \hat{f} . In this case we shall have $E = (g_{j^*}^k)^{-1}(|E_{s_{j^*}, t_{j^*}}|) \subset P_k$.

From all this, we conclude that there is a map $g^* : P_k \rightarrow |K|$ that extends the composition $\hat{f} \circ g_{j^*}^k |E : E \rightarrow |K|$. Since $j = s_{j^*} \leq j^* \leq k$, and $(g_{s_{j^*}}^{j^*})^{-1}(D_{s_{j^*}, t_{j^*}}) \subset |E_{s_{j^*}, t_{j^*}}|$, then $(g_j^k)^{-1}(D) = (g_{s_{j^*}}^k)^{-1}(D_{s_{j^*}, t_{j^*}}) \subset E$.

Therefore according to the first paragraph of this proof, it is simply a matter of showing that $\hat{f} \circ g_{j*}^k |(g_{s_{j*}}^k)^{-1}(D_{s_{j*}, t_{j*}}) : (g_{s_{j*}}^k)^{-1}(D_{s_{j*}, t_{j*}}) \rightarrow |K|$ is homotopic to the map $f \circ g_{s_{j*}}^k |(g_{s_{j*}}^k)^{-1}(D_{s_{j*}, t_{j*}}) : (g_{s_{j*}}^k)^{-1}(D_{s_{j*}, t_{j*}}) \rightarrow |K|$. Since $g_{s_{j*}}^k = g_{s_{j*}}^{j*} \circ g_{j*}^k$, then this comes to showing that $\hat{f} |(g_{s_{j*}}^{j*})^{-1}(D_{s_{j*}, t_{j*}}) : (g_{s_{j*}}^{j*})^{-1}(D_{s_{j*}, t_{j*}}) \rightarrow |K|$ is homotopic to

$$f \circ g_{s_{j*}}^{j*} |(g_{s_{j*}}^{j*})^{-1}(D_{s_{j*}, t_{j*}}) : (g_{s_{j*}}^{j*})^{-1}(D_{s_{j*}, t_{j*}}) \rightarrow |K|.$$

Recall that $\hat{f} = \psi \circ \hat{f}_{s_{j*}, t_{j*}}$, and $\hat{f}_{s_{j*}, t_{j*}} |(g_{s_{j*}}^{j*})^{-1}(D_{s_{j*}, t_{j*}}) = f_{s_{j*}, t_{j*}} \circ g_{s_{j*}}^{j*} |(g_{s_{j*}}^{j*})^{-1}(D_{s_{j*}, t_{j*}})$. But $f_{s_{j*}, t_{j*}} = f^* = \phi \circ f_0 = \phi \circ f | D_{s_{j*}, t_{j*}}$. Since ϕ and ψ are inverse homeomorphisms, then we get the result by substitution. \square

7. ADJUSTMENTS OF AN INDUCED INVERSE SEQUENCE

We want to increase the flexibility of our work so that it can have multiple applications. The major development of this section towards that goal is Lemma 7.8. For the remainder of this section, let X be a nonempty metrizable compactum, $\mathbf{Z} = (|T_j|, g_j^{j+1})$ an induced inverse sequence for X , and $Z = \lim \mathbf{Z}$. Fix $n \geq 0$, including $n = \infty$, and for each $j \in \mathbb{N}$, let $M_j \subset |T_j| \subset I^{n_j}$ be a nonempty closed subset such that $g_j^{j+1}(M_{j+1}) \subset M_j$. For each i , we shall write

- (*1) $\hat{g}_i^k = g_i^k | |T_k^{(n)}| : |T_k^{(n)}| \rightarrow |T_i^{(n)}|$ whenever $i \leq k$,
- (*2) $\hat{\varphi}_i = \varphi_i | |T_i^{(n)}| : |T_i^{(n)}| \rightarrow |\tilde{T}_i^{(n)}|$,
- (*3) $\hat{f}_i^{i+1} = f_i^{i+1} | |\tilde{T}_{i+1}^{(n)}| : |\tilde{T}_{i+1}^{(n)}| \rightarrow |T_i^{(n)}|$, and
- (*4) α_i for the inclusion $|\tilde{T}_i^{(n)}| \hookrightarrow |\tilde{T}_i^{(n+1)}|$.

Of course when $i < k$, then g_i^k factors as $g_i^s \circ g_s^k$ whenever $i \leq s \leq k$. Such a property is “inherited” by \hat{g}_i^k . Let us make some notes about this and others that can be gleaned from the preceding. For each $i < k$,

- (*5) $\hat{g}_i^k = \hat{g}_i^{k-1} \circ \hat{g}_{k-1}^k$,
- (*6) $\hat{g}_{k-1}^k = \hat{f}_{k-1}^k \circ \hat{\varphi}_k$, and
- (*7) $\hat{g}_i^k = \hat{g}_i^{k-1} \circ \hat{f}_{k-1}^k \circ \hat{\varphi}_k$.

We need to name some sets.

Definition 7.1. For each $x \in X$ and $j \in \mathbb{N}$, let $v_{x,j}$ be the vertex of \tilde{T}_j in Lemma 5.2(5), and define:

- (1) $B_{x,j} = \overline{N}(p_{n_j, \infty}(x), 2\delta_j) \cap M_j$,
- (2) $B_{x,j}^+ = \overline{\text{st}}(v_{x,j}, \tilde{T}_j) \cap M_j$,
- (3) $B_{x,j}^\# = \overline{N}(p_{n_j, \infty}(x), \epsilon_j) \cap M_j$, and
- (4) $D_{x,j} = \overline{\text{st}}(v_{x,j}, \tilde{T}_j) \cap |\tilde{T}_j^{(n)}|$.

Notice that \tilde{T}_j induces a unique triangulation $\tilde{L}_{x,j}$ of $\overline{\text{st}}(v_{x,j}, \tilde{T}_j)$. So one has, from Definition 7.1(4), that $D_{x,j} = |\tilde{L}_{x,j}^{(n)}|$. Since $\overline{\text{st}}(v_{x,j}, \tilde{T}_j)$ is contractible and $\dim \tilde{L}_{x,j}^{(n)} \leq n$, then $D_{x,j}$ contracts to a point in $|\tilde{L}_{x,j}^{(n+1)}| \subset |\tilde{L}_{x,j}|$. On the other hand, since T_j is a subdivision of \tilde{T}_j , it follows that T_j induces a unique triangulation $L_{x,j}$ of $\overline{\text{st}}(v_{x,j}, \tilde{T}_j)$, and $|\tilde{L}_{x,j}^{(n)}| \subset |L_{x,j}^{(n)}|$. So if

$M_j = |T_j^{(n)}|$, then $B_{x,j}^+ = |L_{x,j}^{(n)}|$, and $D_{x,j} \subset B_{x,j}^+$. Since $\varphi_j : |T_j| \rightarrow |\tilde{T}_j|$ in Lemma 5.2 is a simplicial approximation to the identity on $|T_j| = |\tilde{T}_j|$, we get that $\hat{\varphi}_j(|T_j^{(n)}|) = |\tilde{T}_j^{(n)}|$ and $\hat{\varphi}_j(B_{x,j}^+) = D_{x,j}$. Another fact of importance to us is that since $\text{st}(v_{x,j}, \tilde{T}_j)$ is connected, then both $B_{x,j}^+$ and $D_{x,j}$ are nonempty metrizable continua. Let us record the preceding information now.

Lemma 7.2. *Let $B_{x,j}^+$ and $D_{x,j}$ be as in Definition 7.1(2, 4) where $M_j = |T_j^{(n)}|$. Then*

- (1) $B_{x,j}^+ = |L_{x,j}^{(n)}|$ and $D_{x,j} = |\tilde{L}_{x,j}^{(n)}|$, $L_{x,j}$ being a subcomplex of T_j and $\tilde{L}_{x,j}$ being a subcomplex of \tilde{T}_j ,
- (2) $B_{x,j}^+$ and $D_{x,j}$ are nonempty metrizable continua,
- (3) $D_{x,j} \subset B_{x,j}^+$,
- (4) $\hat{\varphi}_j(|T_j^{(n)}|) = |\tilde{T}_j^{(n)}|$,
- (5) $\hat{\varphi}_j(B_{x,j}^+) = D_{x,j}$, and
- (6) $D_{x,j}$ contracts to a point in $|\tilde{L}_{x,j}^{(n+1)}|$. □

Here is the definition that will provide the flexibility that we mentioned above.

Definition 7.3. *Suppose that (j_i) is an increasing sequence in \mathbb{N} and for each i , we are given a map $h_i^{i+1} : M_{j_{i+1}} \rightarrow M_{j_i}$ such that h_i^{i+1} is a T_{j_i} -modification of the restriction $g_{j_i}^{j_{i+1}}|_{M_{j_{i+1}}} : M_{j_{i+1}} \rightarrow M_{j_i}$. Then we shall refer to $\mathbf{M} = (M_{j_i}, h_i^{i+1})$ as an **adjustment** of \mathbf{Z} .*

As a consequence of Definition 7.3, $(*)_1$, Lemma 5.2(8), and Lemma 2.2, we obtain a statement about adjustments.

Lemma 7.4. *Let (j_i) be an increasing sequence in \mathbb{N} . Then,*

- (1) $\mathbf{M} = (|T_{j_i}^{(n)}|, \hat{g}_{j_i}^{j_{i+1}})$ is an adjustment of \mathbf{Z} , and
- (2) if $(|T_{j_i}^{(n)}|, h_i^{i+1})$ is an adjustment of \mathbf{Z} , then for all i , $h_i^{i+1} \simeq \hat{g}_{j_i}^{j_{i+1}}$.

Lemma 7.5. *If we take $n = \infty$ and $j_i = i$ for all i , then the adjustment $\mathbf{M} = (|T_i^{(n)}|, \hat{g}_i^{i+1}) = (|T_i|, g_i^{i+1})$ of Lemma 7.4(1) equals \mathbf{Z} . □*

Definition 7.6. *We shall call the adjustment of \mathbf{Z} coming from Lemma 7.5 the **trivial adjustment** of \mathbf{Z} .*

This way we can create a theory of adjustments that takes into account \mathbf{Z} and all other adjustments. The fundamentals for this are in Lemma 7.8.

Definition 7.7. *Let (j_i) be an increasing sequence in \mathbb{N} , $\mathbf{M} = (M_{j_i}, h_i^{i+1})$ be an adjustment of \mathbf{Z} , and $M = \lim \mathbf{M}$. Whenever a point $w = (a_1, a_2, \dots) \in M$, then (a_i) is a sequence in I^∞ which we shall call the sequence **associated with** w . For each $i \in \mathbb{N}$, we get a function $\pi_i : M \rightarrow I^\infty$ by setting $\pi_i(w) = a_i$.*

We shall now give a form of Lemma 3.1 of [AJR] that is suited to our present situation. In Lemma 7.8, we shall use the notation from Definition 7.1. It is worth mentioning that this lemma is independent of the choice of the fixed $n \geq 0$, even $n = \infty$.

Lemma 7.8. *Let (j_i) be an increasing sequence in \mathbb{N} , $\mathbf{M} = (M_{j_i}, h_i^{i+1})$ be an adjustment of \mathbf{Z} , and $M = \lim \mathbf{M}$. Then,*

- (1) *for each $w = (a_1, a_2, \dots) \in M$, the sequence (a_i) in I^∞ associated with w is a Cauchy sequence in I^∞ whose limit lies in X ,*
- (2) *the sequence (π_i) , $\pi_i : M \rightarrow I^\infty$, is a Cauchy sequence of maps whose limit $\pi : M \rightarrow I^\infty$ is a map having the property that $\pi(M) \subset X$,*
- (3) *for each $x \in X$ and $i \in \mathbb{N}$, $B_{x,j_i} \subset B_{x,j_i}^+ \subset B_{x,j_i}^\#$, and $h_i^{i+1}(B_{x,j_{i+1}}^\#) \subset B_{x,j_i}$,*
- (4) *if for each $x \in X$, we define $\mathbf{M}_x = (B_{x,j_i}, h_i^{i+1}|_{B_{x,j_{i+1}}})$, $\mathbf{M}_x^+ = (B_{x,j_i}^+, h_i^{i+1}|_{B_{x,j_{i+1}}^+})$, and $\mathbf{M}_x^\# = (B_{x,j_i}^\#, h_i^{i+1}|_{B_{x,j_{i+1}}^\#})$, then each of \mathbf{M}_x , \mathbf{M}_x^+ , and $\mathbf{M}_x^\#$ is an inverse sequence of metrizable compacta,*
- (5) *for all $x \in X$, $\lim \mathbf{M}_x = \lim \mathbf{M}_x^+ = \lim \mathbf{M}_x^\#$,*
- (6) *for all $x \in X$, $\pi^{-1}(x) = \lim \mathbf{M}_x$, and*
- (7) *if for all $i \in \mathbb{N}$, $|T_{j_i}^{(0)}| \subset M_{j_i}$, then $\pi : M \rightarrow X$ is surjective.*

Proof. For each $i \in \mathbb{N}$, put $m_i = n_{j_i}$. Our choice of metric for I^∞ shows that,

$$(\dagger_1) \text{ for each } i \in \mathbb{N} \text{ and } x \in I^\infty, \rho(p_{m_i, \infty}(x), x) \leq 2^{-m_i}.$$

Let $u \in M_{j_{i+1}} \subset |T|_{j_{i+1}} \subset I^{m_{i+1}}$. Applying Lemma 5.2(8), there is a simplex σ of T_{j_i} , and a face τ of σ such that $p_{m_i}^{m_{i+1}}(u) \in \text{int } \sigma$ and $g_{j_i}^{j_{i+1}}(u) \in \text{int } \tau$. By Definition 7.3, $h_i^{i+1}(u) \in \tau$. Hence $\{h_i^{i+1}(u), p_{m_i}^{m_{i+1}}(u)\} \subset \sigma$. It follows from this and Lemma 5.2(3,4) that,

$$(\dagger_2) \text{ for each } i \in \mathbb{N} \text{ and } u \in M_{j_{i+1}}, \rho(h_i^{i+1}(u), p_{m_i}^{m_{i+1}}(u)) < \frac{\delta_{j_i}}{2} < 2^{-m_i}.$$

Now let $w = (a_1, a_2, \dots) \in M$. Thus, $\rho(a_i, a_{i+1}) = \rho(h_i^{i+1}(a_{i+1}), a_{i+1})$. We know that $a_{i+1} \in M_{j_{i+1}}$. So an application of (\dagger_2) gives us,

$$(\dagger_3) \text{ for all } w = (a_1, a_2, \dots) \in M, \rho(h_i^{i+1}(a_{i+1}), p_{m_i}^{m_{i+1}}(a_{i+1})) < 2^{-m_i}.$$

If one applies the triangle inequality and uses (\dagger_3) and (\dagger_1) with $x = a_{i+1}$, one gets that $\rho(a_i, a_{i+1}) = \rho(h_i^{i+1}(a_{i+1}), a_{i+1}) \leq \rho(h_i^{i+1}(a_{i+1}), p_{m_i}^{m_{i+1}}(a_{i+1})) + \rho(p_{m_i}^{m_{i+1}}(a_{i+1}), a_{i+1}) < 2^{-m_i} + 2^{-m_i} = 2^{1-m_i}$ independently of the choice of $w \in M$. We record this fact:

$$(\dagger_4) \text{ Whenever } w = (a_1, a_2, \dots) \in M \text{ and } i \in \mathbb{N}, \text{ one has that } \rho(a_i, a_{i+1}) < 2^{1-m_i}.$$

Thus (a_i) is a Cauchy sequence in I^∞ , and (π_i) is a Cauchy sequence of maps of M to I^∞ whose limit π is a map of M to I^∞ . But for each i , $a_i \in M_{j_i} \subset |T_{j_i}| \subset |T_{j_i}| \times I_{m_i}^\infty$, so an application of Corollary 3.2 yields that $\pi(M) \subset X$. We have established (1) and (2).

Let $x \in X$. The first part of (3) comes from Lemma 5.2(7). Let $u \in B_{x,j_{i+1}}^\#$. Then, it is true that $\rho(h_i^{i+1}(u), p_{m_i, \infty}(x)) \leq \rho(h_i^{i+1}(u), p_{m_i}^{m_{i+1}}(u)) + \rho(p_{m_i}^{m_{i+1}}(u), p_{m_i, \infty}(x)) = \rho(h_i^{i+1}(u), p_{m_i}^{m_{i+1}}(u)) + \rho(p_{m_i}^{m_{i+1}}(u), p_{m_i}^{m_{i+1}} \circ p_{m_{i+1}, \infty}(x))$. In (\dagger_2) we recorded that $\rho(h_i^{i+1}(u), p_{m_i}^{m_{i+1}}(u)) < \frac{\delta_{j_i}}{2}$. But $u \in B_{x,j_{i+1}}^\#$ implies that $\rho(u, p_{m_{i+1}, \infty}(x)) < \epsilon_{j_{i+1}}$. It follows from Lemma 5.2(1) that $\rho(p_{m_i}^{m_{i+1}}(u), p_{m_i}^{m_{i+1}} \circ p_{m_{i+1}, \infty}(x)) < \delta_{j_i}$. We therefore conclude that $h_i^{i+1}(u) \in \overline{N}(p_{m_i, \infty}(x), 2\delta_{j_i})$, so the second part of (3) is substantiated. From (3), both (4) and (5) follow handily. We must prove (6).

Suppose that $(a_1, a_2, \dots) \in \lim \mathbf{M}_x$ and $i \in \mathbb{N}$. Then $a_i \in B_{x,j_i}$, so $\rho(a_i, p_{m_i, \infty}(x)) \leq 2\delta_{j_i}$. If we apply this, (\dagger_1) , and Lemma 5.2(3), we conclude

that $\rho(a_i, x) \leq \rho(a_i, p_{m_i, \infty}(x)) + \rho(p_{m_i, \infty}(x), x) \leq 2\delta_{j_i} + 2^{-m_i} < 2^{2-m_i} + 2^{-m_i}$. Therefore, $\pi((a_i)) = \lim(a_i) = x$, so we have shown that $\lim \mathbf{M}_x \subset \pi^{-1}(x)$. We have to establish the opposite inclusion.

Suppose that a thread (a_1, a_2, \dots) of \mathbf{M} lies in $\pi^{-1}(x)$. For the next, make use of (\dagger_1) , (\dagger_4) , and Lemma 5.2(2). For all $i \in \mathbb{N}$, $\rho(a_i, p_{m_i, \infty}(x)) \leq \rho(a_i, x) + \rho(x, p_{m_i, \infty}(x)) \leq \sum_{k=i}^{\infty} \rho(a_k, a_{k+1}) + 2^{-m_i} \leq \sum_{k=i}^{\infty} 2^{1-m_k} + 2^{-m_i} \leq 2 \cdot 2^{1-m_i} + 2^{-m_i} = 5 \cdot 2^{-m_i} < \epsilon_{m_i}$. This puts $a_i \in B_{x, i}^{\#}$. So, $(a_1, a_2, \dots) \in \lim \mathbf{M}_x^{\#} = \lim \mathbf{M}_x$, as required to complete the proof of (6).

To prove (7), we only need to show that for each $x \in X$, $\pi^{-1}(x) \neq \emptyset$, and for this we use (6). It is sufficient to demonstrate that for each $i \in \mathbb{N}$, $B_{x, j_i} \neq \emptyset$. Lemma 5.2(4) states that $\text{mesh } T_{j_i} < \frac{\delta_{j_i}}{2}$. So $\overline{N}(p_{m_i, \infty}(x), 2\delta_{j_i})$ has to contain some $v \in T_{j_i}^{(0)}$. By hypothesis, $v \in M_{j_i}$. So $B_{x, j_i} \neq \emptyset$. \square

More technical facts need to be established.

Lemma 7.9. *Let (j_i) be an increasing sequence in \mathbb{N} and $\mathbf{M} = (|T_{j_i}^{(n)}|, \hat{g}_{j_i}^{j_i+1})$ be the adjustment of \mathbf{Z} as indicated in Lemma 7.4(1). Then for each $i < k$ in \mathbb{N} and $x \in X$,*

- (1) $\hat{g}_{j_i}^{j_k} = \hat{g}_{j_i}^{j_{k-1}} \circ \hat{g}_{j_{k-1}}^{j_k}$,
- (2) $\hat{g}_{j_{k-1}}^{j_k} = \hat{f}_{j_{k-1}}^{j_k} \circ \hat{\varphi}_{j_k}$,
- (3) $\hat{g}_{j_i}^{j_k} = \hat{g}_{j_i}^{j_{k-1}} \circ \hat{f}_{j_{k-1}}^{j_k} \circ \hat{\varphi}_{j_k}$,
- (4) $\hat{g}_{j_i}^{j_k}(B_{x, j_k}^+) \subset B_{x, j_i}^+$, and
- (5) $\hat{f}_{j_{k-1}}^{j_k}(D_{x, j_k}) \subset B_{x, j_{k-1}}^+$.

Proof. We get (1)-(3) from $(*)_5$ -($*$ ₇) and (4) from Lemma 7.8(4) when applying the adjustment \mathbf{M} to the inverse sequence \mathbf{M}_x^+ . One arrives at (5) from Lemma 7.2(5), (2), and (4) as applied to $\hat{g}_{j_{k-1}}^{j_k}$. \square

Lemma 7.10. *Let $(|T_{j_i}^{(n)}|, h_i^{i+1})$ be an adjustment of \mathbf{Z} , $x \in X$, and $i \in \mathbb{N}$. Then,*

- (1) $h_i^{i+1}(B_{x, j_{i+1}}^+) \subset B_{x, j_i}^+$, and
- (2) $h_i^{i+1}|B_{x, j_{i+1}}^+ \simeq \hat{g}_{j_i}^{j_{i+1}}|B_{x, j_{i+1}}^+$ as maps to B_{x, j_i}^+ .

Proof. Lemma 7.8(4) implies (1). By Lemma 7.2(1), $B_{x, j_i}^+ = |L_{x, j_i}^{(n)}|$ where L_{x, j_i} is a subcomplex of T_{j_i} . By this, Lemma 7.9(4), the fact that h_i^{i+1} is a T_{j_i} -modification of $\hat{g}_{j_i}^{j_{i+1}}$, and Lemma 2.2 we get (2). \square

Lemma 7.11. *Let (j_i) be an increasing sequence in \mathbb{N} and $\mathbf{M} = (|T_{j_i}^{(n)}|, \hat{g}_{j_i}^{j_i+1})$ be the adjustment of \mathbf{Z} as indicated in Lemma 7.4(1). Then for each $i < k$ in \mathbb{N} and $x \in X$,*

- (1) $\alpha_{j_k}(D_{x, j_k}) \subset |\tilde{L}_{x, j_k}^{(n+1)}|$,
- (2) $\varphi_{j_k}(|L_{x, j_k}^{(n+1)}|) = |\tilde{L}_{x, j_k}^{(n+1)}|$,
- (3) $g_{j_i}^{j_k}(B_{x, j_k}^+) \subset |L_{x, j_i}^{(n+1)}|$,
- (4) $g_{j_i}^{j_k}|B_{x, j_k}^+ : B_{x, j_k}^+ \rightarrow |L_{x, j_i}^{(n+1)}|$ is homotopic to a constant map, and
- (5) in particular if $n = \infty$, then $g_{j_i}^{j_k}|B_{x, j_k}^+ : B_{x, j_k}^+ \rightarrow B_{x, j_i}^+$ is homotopic to a constant map.

Proof. We obtain (1) from the triangulation of D_{x,j_k} from Lemma 7.2(1) and $(*_4)$. Item (2) comes from Lemma 7.2(5) with n replaced by $n + 1$. To get (3), use Lemma 7.9(4), Lemma 7.2(1) and the fact that $|L_{x,j_i}^{(n)}| \subset |L_{x,j_i}^{(n+1)}|$. To obtain (4), let us write the map $g_{j_i}^{j_k}$ as a composition. Let $t \in B_{x,j_k}^+$. Then $g_{j_i}^{j_k}(t) = g_{j_i}^{j_k-1} \circ f_{j_k-1}^{j_k} \circ \alpha_{j_k} \circ \widehat{\varphi}_{j_k}(t)$ because of Lemma 7.9(3) and $(*_4)$.

But Lemma 7.2(5) shows that $\widehat{\varphi}_{j_k}(B_{x,j_k}^+) = D_{x,j_k} = |\widetilde{L}_{x,j_k}^{(n)}|$ which contracts to a point in $|\widetilde{L}_{x,j_k}^{(n+1)}|$ by (6) of that lemma; so $\alpha_{j_k} : D_{x,j_k} \rightarrow |\widetilde{L}_{x,j_k}^{(n+1)}|$ is homotopic to a constant map. Now $f_{j_k-1}^{j_k}(|\widetilde{L}_{x,j_k}^{(n+1)}|) \subset |L_{x,j_k-1}^{(n+1)}|$ as a result of applying Lemma 7.9(5) in the case that n is replaced by $n + 1$. Next apply Lemma 7.9(3) to complete the argument for (4). The statement (5) follows when we choose $n = \infty$ in (4). \square

Lemma 7.12. *Let $\mathbf{M} = (|T_{j_i}^{(n)}|, h_i^{i+1})$ be an adjustment of \mathbf{Z} , $M = \lim \mathbf{M}$, and $\pi : M \rightarrow X$ the map of Lemma 7.8(2). Then for each $x \in X$, $\pi^{-1}(x) \neq \emptyset$, i.e., π is surjective.*

Proof. Since for each i , $|T_{j_i}^{(0)}| \subset |T_{j_i}^{(n)}|$, then Lemma 7.8(7) yields that $\pi^{-1}(x) \neq \emptyset$. \square

8. CELL-LIKE MAP

Theorem 8.5 is the third and last step in showing how we can obtain from a given metrizable compactum X , a metrizable compactum Z that is a substitute for X in the sense of extension theory, where the replacement Z is rich in the (useful) properties seen in Lemma 5.2.

Definition 8.1. *A compact metrizable space is said to have **trivial shape** (or is called **cell-like**) if it has the shape of a point [MS]. Such a space always has to be a nonempty continuum.*

Lemma 8.2. *Let $\mathbf{X} = (X_i, h_i^{i+1})$ be an inverse sequence of metrizable compacta and $X = \lim \mathbf{X}$. Then X has trivial shape if for each $i \in \mathbb{N}$, there exists $j > i$ such that $h_i^j : X_j \rightarrow X_i$ is null homotopic.* \square

Definition 8.3. *A proper map of one space to another whose fibers are cell-like will be called a **cell-like map**.*

One should note that cell-like maps have to be surjective. Since every finite-dimensional metrizable compactum embeds in some \mathbb{R}^n , then one can use Corollary 5A, p. 145 of [Da], to justify the next fact.³

Lemma 8.4. *If X is a finite-dimensional cell-like space, then there exists $n \in \mathbb{N}$ so that X can be embedded in \mathbb{R}^n as a cellular subset.* \square

Theorem 8.5. *Let X be a nonempty metrizable compactum, $\mathbf{Z} = (|T_i|, g_i^{i+1})$ an induced inverse sequence for X , and $Z = \lim \mathbf{Z}$. Then the map $\pi : Z \rightarrow X$ of Lemma 7.8(2) under the trivial adjustment is a cell-like map from the metrizable compactum $Z = \lim \mathbf{Z}$. Moreover,*

(*) *for each CW-complex K_0 with $X \tau K_0$, $Z \tau K_0$.*

³This particular property of cell-like continua plays no role herein.

Proof. By Definition 7.6, $\mathbf{Z} = (|T_i|, g_i^{i+1})$ is the trivial adjustment of \mathbf{Z} . By Lemma 7.12, π is surjective. Fix $x \in X$. From Lemma 7.8(5,6), $\pi^{-1}(x) = \lim \mathbf{M}_x^+$. In this case, $\mathbf{M}_x^+ = (B_{x,i}^+, g_i^{i+1}|B_{x,i+1}^+)$. By Lemma 7.11(5), every bonding map $g_i^{i+1}|B_{x,i+1}^+ : B_{x,i+1}^+ \rightarrow B_{x,i}^+$ in \mathbf{M}_x^+ is homotopic to a constant map. So Lemma 8.2 may be used to complete the proof that π is cell-like. Apply Theorem 6.1 to obtain (*). \square

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